Note on rainbow connection number of dense graphs*

Jiuying Dong, Xueliang Li

Center for Combinatorics and LPMC-TJKLC

Nankai University, Tianjin 300071, China

Email: jiuyingdong@126.com; lxl@nankai.edu.cn

Abstract

An edge-colored graph G is rainbow connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection number of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. Following an idea of Caro et al., in this paper we also investigate the rainbow connection number of dense graphs. We show that for $k \geq 2$, if G is a non-complete graph of order n with minimum degree $\delta(G) \geq \frac{n}{2} - 1 + log_k n$, or minimum degree-sum $\sigma_2(G) \geq n - 2 + 2log_k n$, then $rc(G) \leq k$; if G is a graph of order n with diameter 2 and $\delta(G) \geq 2(1 + log_{\frac{k^2}{3k-2}} k) log_k n$, then $rc(G) \leq k$. We also show that if G is a non-complete bipartite graph of order n and any two vertices in the same vertex class have at least $2log_{\frac{k^2}{3k-2}} k log_k n$ common neighbors in the other class, then $rc(G) \leq k$.

Keywords: rainbow coloring, rainbow connection number, parameter $\sigma_2(G)$

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1 Introduction

All graphs under our consideration are finite, undirected and simple. For notation and terminology not defined here, we refer to [1]. Let G be a graph. The length of a path in G is the number of edges of the path. The distance between two vertices u and v in G, denoted by $d(u, v)$, is the length of a shortest path connecting them in G. If there

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is no path connecting u and v, we set $d(x, y) := \infty$. An edge-coloring of a graph is a function from its edges set to the set of natural numbers. A graph G is rainbow edge-connected if for every pair of distinct vertices u and v of G, G has a $u - v$ path whose edges are colored with distinct colors. This concept was introduced by Chartrand et al. [4]. The minimum number of colors required to rainbow color a connected graph is called its rainbow connection number, denoted by $rc(G)$. Observe that if G has n vertices, then $rc(G) \leq n - 1$. Clearly, $rc(G) \geq diam(G)$, the diameter of G. In [4], Chartrand et al. determined the rainbow connection numbers of wheels, complete graphs and all complete multipartite graphs. In [3], Chakraborty et al. proved that given a graph G, deciding if $rc(G) = 2$ is NP-Complete. In particular, computing $rc(G)$ is NP-Hard.

If $\delta(G) \geq \frac{n}{2}$, then $diam(G) = 2$, but we do not know if this guarantees $rc(G) = 2$. In [2], Caro et al. investigated the rainbow connection number of dense graphs, and they got the following results.

Theorem 1.1. *Any non-complete graph with $\delta(G) \geq \frac{n}{2} + logn$ has $rc(G) = 2$.*

Theorem 1.2. *Let $c = \frac{1}{log(9/7)}$. If G is a non-complete bipartite graph with n vertices and any two vertices in the same vertex class have at least $2clogn$ common neighbors in the other class, then $rc(G) = 3$.*

We will follow their idea to investigate dense graphs again. And we get the following results.

Theorem 1.3. *Let $k \geq 2$ be an integer. If G is a non-complete graph of order n with $\delta(G) \geq \frac{n}{2} - 1 + log_k n$, then $rc(G) \leq k$.*

Theorem 1.4. *Let $k \geq 2$ be an integer. If G is a non-complete graph of order n with $\sigma_2(G) \geq n - 2 + 2log_k n$, then $rc(G) \leq k$.*

Theorem 1.5. *Let $k \geq 3$ be an integer. If G is a non-complete bipartite graph of order n and any two vertices in the same vertex class have at least $2log_{\frac{k^2}{3k-2}} klog_k n$ common neighbors in the other class, then $rc(G) \leq k$.*

In [3], Chakraborty et al. showed the following result.

Theorem 1.6. *If G is a graph of order n with diameter 2 and $\delta(G) \geq 8logn$, then $rc(G) \leq 3$. Furthermore, such a coloring is given with high probability by a uniformly random 3-edge-coloring of the graph G, and can also be found by a polynomial time deterministic algorithm.*

Now we get the following result.

Theorem 1.7. *Let $k \geq 3$ be an integer. If G is a graph of order n with diameter 2 and $\delta(G) \geq 2(1 + log_{\frac{k^2}{3k-2}} k)log_k n$, then $rc(G) \leq k$.*

2 Proof of the theorems

Proof of Theorem 1.3: Let G be a non-complete graph of order n with $\delta(G) \geq \frac{n}{2} - 1 + log_k n$. We use k different colors to randomly color every edge of G. In the following we will show that with positive probability, such a random coloring make G rainbow connected. For any pair $u, v \in V(G), uv \notin E(G)$, since $d(u) \geq \frac{n}{2} - 1 + log_k n$, $d(v) \geq \frac{n}{2} - 1 + log_k n$, there are at least $2log_k n$ common neighbors between u and v, that is $|N(u) \cap N(v)| \geq 2log_k n$. Hence there are at least $2log_k n$ edge-disjoint paths of length two from u to v. For any $w \in N(u) \cap N(v)$, the probability that the path uwv is not a rainbow path is $\frac{1}{k}$. Hence, the probability that all these edge-disjoint paths are not rainbow is at most $(\frac{1}{k})^{2log_k n} = \frac{1}{n^2}$. Since there are less than $\binom{n}{2}$ pairs non-adjacent vertices, and $\binom{n}{2}\frac{1}{n^2} < 1$. We may get that with positive probability, each pair of non-adjacent vertices are connected by a rainbow path. This completes the proof of Theorem 1.3.

Proof of Theorem 1.4: Let G be a non-complete graph of order n with $\sigma_2(G) \geq n - 2 + 2log_k n$. We use k different colors to randomly color every edge of G. In the following we will show that with positive probability, such a random coloring make G rainbow connected. For any pair $u, v \in V(G), uv \notin E(G)$, as $\sigma_2(G) \geq n - 2 + 2log_k n$, it follows that $|N(u) \cap N(v)| \geq 2log_k n$. Similar to the proof of Theorem 1.3, we may get that with positive probability, each pair of non-adjacent vertices are connected by a rainbow path. This completes the proof of Theorem 1.4.

Proof of Theorem 1.5: Let G be a non-complete bipartite graph of order n and any two vertices in the same vertex class have at least $2log_{\frac{k^2}{3k-2}} klog_k n$ common neighbors in the other class. We use k different colors to randomly color every edge of G. In the following we will show that with positive probability, such a random coloring make G rainbow connected. For every pair $u, v \in V(G)$ and u, v are in the same class of $V(G)$, then the distance of $d(u, v) = 2$, as $|N(u) \cap N(v)| \geq 2log_{\frac{k^2}{3k-2}} klog_k n$, there are at least $2log_{\frac{k^2}{3k-2}} klog_k n$ edge-disjoint paths of length two from u to v. The probability that all these edge-disjoint paths are not rainbow is at most $(\frac{1}{k})^{2log_{\frac{k^2}{3k-2}} klog_k n} < (\frac{1}{k})^{2log_k n} = \frac{1}{n^2}$. For every pair $u, v \in V(G)$ from different classes of G and $uv \notin E(G)$, then the distance of $d(u, v)$ is 3. Fix a neighbor w_u of u, for any $u_i \in N(w_u) \cap N(v)$, the probability that $uw_u u_i v$ is not a rainbow path is $\frac{3k-2}{k^2}$. We know $|N(w_u) \cap N(v)| \geq 2log_{\frac{k^2}{3k-2}} klog_k n$. Hence, the probability that all these edge-disjoint paths are not rainbow is at most $(\frac{3k-2}{k^2})^{2log_{\frac{k^2}{3k-2}} klog_k n} = \frac{1}{n^2}$. Thus, we may get that with positive probability, each pair of non-adjacent vertices are connected by a rainbow path. This completes the proof of Theorem 1.5.

Proof of Theorem 1.7:

Let G be a graph of order n with diameter 2. We use k different colors to randomly color every edge of G. In the following we will show that with positive probability, such a random coloring make G rainbow connected. For any two non-adjacent vertices u, v, if $|N(u) \cap N(v)| \geq 2log_k n$, then there are at least $2log_k n$ edge-disjoint paths of length two from u to v. The probability that all these edge-disjoint paths are not rainbow is at most $(\frac{1}{k})^{2log_k n} = \frac{1}{n^2}$. Otherwise, $|N(u) \cap N(v)| < 2log_k n$. Let $A = N(u) \backslash N(v), B = N(v) \backslash N(u)$, then $|A| \geq 2log_{\frac{k^2}{3k-2}} klog_k n, |B| \geq 2log_{\frac{k^2}{3k-2}} klog_k n$. As the diameter of G is two, for any $x \in A$, $\exists y_x \in N(v)$ such that $xy_x \in E(G)$, that is $xy_x v$ is a path of length 2. Now, we will consider the set of at least $2log_{\frac{k^2}{3k-2}} klog_k n$ edge-disjoint paths $P = \{uxy_x v : x \in A\}$. For every $x \in A$, the probability that $uxy_x v$ is not a rainbow path is $\frac{3k-2}{k^2}$. Moreover, this event is independent of the corresponding events for all other members of A, because this probability does not change even with full knowledge of the colors of all edges incident with v. Therefore, the probability that all these edge-disjoint paths are not rainbow is at most $(\frac{3k-2}{k^2})^{2log_{\frac{k^2}{3k-2}} klog_k n} = \frac{1}{n^2}$. Since there are less than $\binom{n}{2}$ pairs non-adjacent vertices, and $\binom{n}{2} \frac{1}{n^2} < 1$. We may get that with positive probability, each pair of non-adjacent vertices are connected by a rainbow path. This completes the proof of Theorem 1.7.

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